OSCILLATIONS OF A CYLINDRICAL SHELL FILLED WITH A FLUID AND SURROUNDED BY AN ELASTIC MEDIUM

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Utilization of exact solutions of three-dimensional nonstationary problems in elasticity for a series of layered media provides a method for the investigation of the singularities of wave propagation in the low frequency range. Recent investigations have dealt with plane, uniformly thick lamina whose bounding surfaces were either free or in contact with elastic or fluid media. As a result of these investigations it has been possible to substantiate and to correct known classical equations concerning the oscillation of thin plates [1] and to obtain the corresponding equations for laminated plates [2]. Moreover, investigation of the exact solutions for an elastic layer bounded by fluids and for a plate on an elastic foundation permitted the study of low frequency wave propagation processes in these media [3 and 4].

The problem below deals with the oscillations of a layer with nonplanar boundaries. It is concerned with wave propagation in a cylindrical shell bounded on the outside by an elastic medium and filled with a fluid. Thus problem is of interest, in particular, in connection with the study of seismic waves in the neighborhood of bore holes and in the isolation of wave interference resulting from the presence of bore holes.

1. Consider a cylindrical coordinate system r, θ, z with a given elastic cylindrical layer $1\,(r_1 < r < r_2)$ surrounded by an elastic medium $2\,(r > r_2)$ and a fluid cylinder $0\,(r < r_1)$. All media are assumed to be homogeneous and isotropic, with the i th medium ($i = 0, 1, 2$) having a density ρ_i and propagation velocities v_{pi}, v_{si} for longitudinal and transverse waves, respectively ($v_{s0} = 0$). The displacement vector in the elastic media satisfies the Lamé equation, while in the fluid it is given by the linear equations of hydroacoustics.

On the boundary of the elastic media, the contact conditions are of two types: (1) rigid contact, whereupon the normal and tangential displacements and stresses are continuous; (2) nonrigid contact, whereupon only the normal stresses and displacements are continous, while the shearing stresses vanish. For the elastic-fluid boundary $r = r_1$, the normal displacements are continuous while the difference in normal stresses and the tangential stress are equal to external loads which define the source. The latter, which takes effect at $t = 0$, is applied to the surface $r = r_1$ and is axisymmetric.

The problem of determining the displacement field consists of solving the equations of Lamé and of linear hydroacoustics with zero initial conditions and with boundary conditions represented by sources, subject to compatibility requirements on the boundaries $r = r_1$ and $r = r_2$, respectively. The solution to this problem, is obtained with the aid of Fourier and Laplace integral transforms. The components of the displacement vector, $u_{ri}(t, z, r)$ and $u_{zi}(t, z, r)$ ($u_{\theta i} \equiv 0$) in the ith medium are given by

$$\left.\begin{matrix} u_{ri} \\ u_{zi} \end{matrix}\right\} = \int_0^\infty \left\{\begin{matrix} \cos kz \\ \sin kz \end{matrix}\right\} \frac{dk}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \left.\begin{matrix} U_{ri}(k, \eta, r) \\ U_{zi}(k, \eta, r) \end{matrix}\right\} \exp(kt\eta v_{s1})\, d\eta \qquad (1.1)$$

where $U_{ri}(k, \eta, r)$ and $U_{zi}(k, \eta, r)$ are linear combinations of cylindrical functions of order zero and one.

The coefficients of this linear combination are obtained from a system of seven algebraic equations based on the boundary conditions. Since the expressions for $U_r(k, \eta, r)$ and $U_z(k, \eta, r)$ are extremely cumbersome, they will not be written here.

2. Before investigating the resultant displacement fields it is necessary to evaluate the inside integrals. For this purpose, a study was made of the singularities of the integrands in the η plane. These functions have an essential singularity at $\eta=\infty$, branch points at $\pm i\gamma_2^{-1}$ and $\pm i\delta_2^{-1}$ ($\gamma_2 = v_{s_1} v_{p_2}^{-1}$, $\delta_2 = v_{s_1} v_{s_2}^{-1}$) and poles in the left half-plane and on the imaginary axis. The location of these poles coincides with the roots of Eq.

$$\Delta(kr_1, kr_2, \eta) = 0 \qquad (2.1)$$

where Δ is the determinant of the above mentioned system of algebraic equations. Since the roots are symmetrically located with respect to the real axis, it is only necessary to study the roots in the upper half-plane.

In investigating (2.1) it is convenient to begin with an examination of the region

$$kr_2 \ll 1 \qquad (2.2)$$

where, for infinite η, the cylindrical functions found in Δ may be replaced by the first terms of their series expansions. The roots of (2.1) under conditions (2.2) are either located at a finite distance from the origin or given by formulas of the form

$$\eta = O\,[(kr_2 - kr_1)^{-1}].$$

In view of this, we will divide the solutions to (2.1) into two classes.

To find the roots of the first class (located at a finite distance) in region (2.2), we may write the following approximate Eqs.:

$$(p\eta^2 + \alpha_0^2)(1 - \gamma_1^2 + \sigma\gamma_1^2) + x(1 - \sigma)(p\eta^2\gamma_1^2 - \alpha_0^2 + \gamma_1^2\alpha_0^2) = 0 \qquad (2.3)$$

$$(A + \sigma\alpha_1^2)(p\eta^2 + \alpha_0^2) + x(1 - \sigma)(p\eta^2\alpha_1^2 - A\alpha_0^2) = 0 \qquad (2.4)$$

corresponding to the case of rigid and nonrigid contact, respectively. The following notation has been introduced in (2.3) and (2.4):

$$\begin{gathered} p = \rho_0\rho_1^{-1}, \qquad \sigma = \mu_2\mu_1^{-1}, \qquad \gamma_0 = v_{s1}v_{p0}^{-1}, \qquad \gamma_1 = v_{s1}v_{s1}^{-1} \\ \mu_1 = \rho_1 v_{s1}^2, \qquad \mu_2 = \rho_2 v_{s2}^2, \qquad x = r_1^2 r_2^{-2}, \qquad \alpha_0^2 = 1 + \gamma_0^2\eta^2 \\ \alpha_1^2 = 1 + \gamma_1^2\eta^2, \qquad A = 3 - 4\gamma^2 + \eta^2(1 - \gamma_1^2) \end{gathered} \qquad (2.5)$$

Simple analysis of (2.3) and (2.4) shows that the first equation is linear in η^2 while the second is quadratic. The roots of both equations lie on the imaginary axis and move monotonously with increasing x. The boundary points for the location of these roots are obtained from (2.3) and (2.4) by setting $x=0$ and $x=1$. Investigations show that for real media (for which $3v_{p0} < V_{p1}$, $\sigma \sim 0.1$, $p \sim 0.1$) the intervals for the roots of (2.4) are

$$\left(\frac{i\sqrt{\sigma}}{\sqrt{p + \sigma\gamma_0^2}},\ \frac{i}{\sqrt{p + \gamma_0^2}}\right), \qquad \left(i\,\frac{\sqrt{3 - 4\gamma_1^2 + \sigma}}{\sqrt{1 - \gamma_1^2 + \sigma\gamma_1^2}},\ i2\sqrt{1 - \gamma_1^2}\right) \qquad (2.6)$$

The root for (2.3), for real media, lies in the first interval in (2.6) and near the root of (2.4) having the smaller modulus.

We now turn to an examination of the movement of roots of (2.1) as kr_1 and kr_2

increase. To investigate the movement of the roots of the first class, it is necessary to include additional terms in the series expansions of the cylindrical functions. The character of the movement of the roots is determined by the relation between the initial position $\mathrm{Im}\{\eta_0\}$ of the root and δ_2^{-1}. If $\mathrm{Im}\{\eta_0\} < \delta_2^{-1}$, then the root remains on the imaginary axis as kr_1 and kr_2 increase; if the opposite is true, then the root lies in the left half-plane.

3. After completing a qualitative investigation of the singularities of the integrands, the Mellin integral may be written as sum of residues and contour integrals enclosing the branch points $\pm i\gamma_2^{-1}$ and $\pm i\delta_2^{-1}$. As a result, the displacement field is expressible as the sum of terms representing interference waves. The Fourier integral of the integrals along the branch cuts characterizes waves which propagate in all media along the z-axis with constant velocities v_{s2} and v_{p2}. These waves are of no particular interest in the study of oscillations in connection with bore holes. The velocities of the remaining interference waves depend on the frequency.

Waves with dispersion are represented by integrals of the type

$$\int_0^\infty F(k, r)\, e^{i[\omega(k)t-kz]} e^{-\alpha(k)t}\, dk \tag{3.1}$$

where the functions $\omega(k)$ and $\alpha(k)$ are given by

$$\omega(k) = kv_{s1}\,\mathrm{Im}\,\eta, \qquad \alpha(k) = kv_{s1}\,|\mathrm{Re}\,\eta| \tag{3.2}$$

with η determining the position of the root in the η plane. The function $F(k, r)$ is easily found as a result of the explicit form of the solution and the functional dependence $\eta(k)$. In the variable of integration k in (3.1) is changed to ω, then (3.1) is replaced by

$$\int_{\omega_0}^\infty F_1(\omega, r)\, e^{i[\omega t - k(\omega)z]} e^{-\beta(\omega)t}\, d\omega \tag{3.3}$$

where $k(\omega)$ is the inverse function of $\omega(k)$, and

$$F_1(\omega, r) = F[k(\omega), r]\, k'(\omega), \qquad \beta(\omega) = \alpha[k(\omega)] \tag{3.4}$$

For $\beta(\omega) \equiv 0$, (3.3) is a Fourier integral representing the superposition of undamped oscillations whose frequency spectrum lies in the interval $[\omega_0, \infty)$. The phase velocities v_φ of these oscillations are given by

$$v_\varphi = v_{s1}\,\mathrm{Im}\,\eta \tag{3.5}$$

If $\beta(\omega) \not\equiv 0$, then (3.3) characterizes a group of exponentially decaying oscillations which propagate with the velocities given in (3.5). For $\beta(\omega)t \ll 1$, the spectrum is approximately determined by $F_1(\omega, r)$ and the oscillation frequencies lie in the interval $[\omega_0, \infty)$.

In accordance with (3.2), the quantity ω_0 in (3.3) is nonzero, only for integrals associated with roots of the first class. Analysis of (2.1) shows that in the case of roots of the second class $\omega_0 \lesssim v_{s1}(r_2 - r_1)^{-1}$. Thus, if interest is confined to the low frequency range

$$\omega \ll v_{s1}(r_2 - r_1)^{-1} \tag{3.6}$$

one need only take into account those expressions in (3.3) which correspond to roots of the first class.

4. We will now study further the displacement field in the region (3.6), where, according to the results of the root investigation, either one or two waves may be observed. These waves have been observed in seismic experiments [5] and are known as water

waves and cylindrical waves. The velocity v_1 of the cylindrical waves for $\omega = 0$ satisfies the inequalities

$$v_{s1}^- < v_1 < v_{s1}^+ \qquad \left(v_{s1}^- = \left(\frac{3 - 4\delta_1^2 + \sigma}{1 - \gamma_1^2 + \sigma\gamma_1^2}\right)^{1/2}, \; v_{s1}^+ = 2v_{s1}\sqrt{1 - \gamma_1^2}\right) \qquad (4.1)$$

Investigation of the lower limit v_{s1}^- shows that it increases, as σ increases within the interval [0, 1], from the velocity of propagation in a rod $v_{s1}\sqrt{(3 - 4\gamma_1^2)(1 - \gamma_1^2)^{-1}}$ to the plate velocity v_{s1}^+. The plate velocity and bar velocity generally differ little from each other, so that the velocity of cylindrical waves varies little as a function of σ, the shear modulus or shell thickness. The cylindrical waves exist even in the absence of media 0 and 2. Hence, the cylindrical waves are related to the oscillations of the shell itself. If the shell is in rigid contact with the elastic medium, then this type of shell oscillation is suppressed.

In contrast with the cylindrical waves, the water waves exist in the case of rigid contact as well. The velocity v_2 of these waves for $\omega = 0$ lies within the interval

$$\frac{v_{s1}\sqrt{\sigma}}{\sqrt{p + \sigma\gamma_0^2}} \leqslant v_2 \leqslant \frac{v_{s1}}{\sqrt{p + \gamma_0^2}} \qquad (4.2)$$

for both types of contact, and, as numerical calculations show, it depends little on the type of contact. However, it varies appreciably with shell thickness. In seismic experiments by Riggs [6], the introduction into the bore hole of even a thin shell ($r_1 r_2^{-1} = 0.97$) increased the velocity of the water waves by 40%. This increase in velocity is confirmed by numerical calculations based on (2.3), (2.4) and (3.5).

Water waves and cylindrical waves can propagate along the z-axis with or without attenuation. The propagation characteristics are related to the location of the roots in the η plane and are determined by the relation between the velocity v_{s2} and the velocities of the water waves and cylindrical waves. The conditions for propagation without exponential decay are: $v_1 < v_{s2}$ and $v_2 < v_{s2}$ for the cylindrical waves and water waves respectively. Under these conditions, the waves in the elastic medium are attenuated exponentially with an increase in the distance from the shell. If the opposite relations are satisfied, then the water waves are dilatational and propagate in all directions. The spectral dependence of the displacements, as in the case of uniformly thick plane layers [3 and 4], displays maximums which shift toward the lower frequencies with an increase in time.

In conclusion, we note that the methods employed here permit also the study of low frequency oscillations for shell problems for which exact solutions may be constructed.

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